MATHS CHALLENGE 2010
SOLUTIONS – PART 2

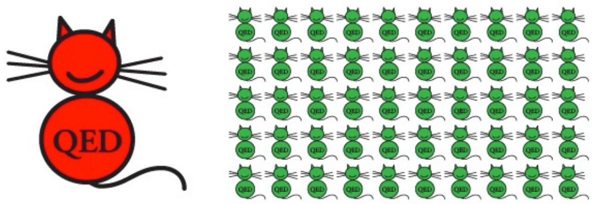
MEDIUM 1

Kylie and Dannii ran a 100 meters race, and Kylie won by 5 meters. They plan to run a second race, with Kylie beginning 5 meters behind the starting line. Assuming both run the same speed as in the first race, who will win?

ANSWER: When Dannii has run 95 meters, Kylie will have run 100 meters and so will have caught up. Since Kylie is running faster, she will beat Dannii over the last 5 meters.

MEDIUM 2

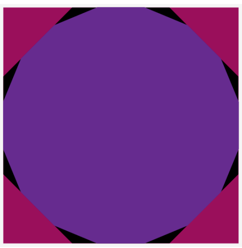
A liter of paint is required to paint a 4 meter high statue of QEDcat. How many liters do you need to paint 50 statuettes of QEDcat, each 20 centimeters high?



ANSWER: Each statuette is 1/20 the height of the statue. That means the area of the each statuette is $1/20^2$ that of the statue, and so will require 1/400 the paint. Since there are 50 statuettes, overall we require 50/400 the paint, or 1/8 of a liter.

MEDIUM 3

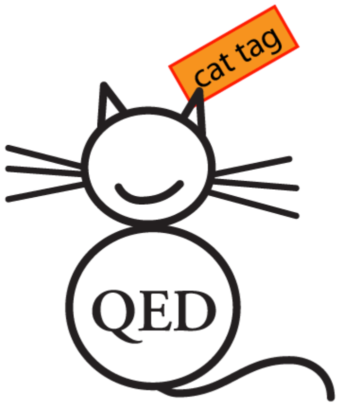
Begin with a square of length 2, and draw an octagon inside it, as pictured. Now draw a regular 16-sided polygon inside the octagon, and so on. What is the area trapped inside all the polygons?



ANSWER: The polygons altogether are trapping a circle. Since each polygon contains the midpoints of the four edges of the surrounding square, the diameter of this circle must be 2. So, the area of the circle must be π.

MEDIUM 4

A forest outside of Melbourne is inhabited by wild QEDcats. One day you catch 30 QEDcats. You tag their ears, and set them free. The next day you catch another 20 QEDcats, of which 2 have tagged ears. How many QEDcats do you estimate inhabit the forest?



ANSWER: Since 10% of the second day's QEDcats were tagged, your best guess is that the previous day you tagged 10% of all the QEDcats. That is, 30 of the beasts amounts to 10% of the whole population. So, you guess that there are 300 QEDcats in the forest.

MEDIUM 5

You have ten large sacks of coins, but one of the sacks contains counterfeit coins. The real coins weigh 20 grams each and the counterfeit coins weigh 15 grams each. You have a scale with which you can weigh any number of coins. Using just one weighing, how can you tell which is the sack of counterfeit coins?

ANSWER: This problem was used to test the TV detective Columbo in the (not very good) episode *The Bye-Bye Sky High IQ Murder Case*. Take 1 coin from the first sack, 2 from the second sack, and so on. Weighing this pile, the number of grams underweight (divided by 5) indicates how many counterfeit coins are in the pile, and so which sack the coins came from.

MEDIUM 6

Replace the letters Q, E, and D by different digits to make the following equation true.

$$QED = (Q + E + D) \times Q \times E \times D$$

ANSWER: This one turned out to be more evil than we intended. Basically, there's nothing to do but go through a lot of cases, the workload is dramatically reduced by the fact that all the digits are to be different. Using divisibility tests also makes life a lot easier:

a) if QED is odd then the digits Q, E and D must all be odd;
b) if QED is divisible by 5 then D = 5;
c) if QED is divisible by 3 then Q + E + D is divisible by 3;
d) if QED is divisible by 9 then Q + E + D is divisible by 9;
e) If QED is divisible by 4 then ED is divisible by 4.
f) If QED is divisible by 7 then QE – (2 x D) and (10 x Q) – (2 x ED) are divisible by 7;

(See our column here for some discussion of these divisibility tricks, and especially divisibility by 7.)

Now, suppose QED is odd and *not* divisible by 3. Then the digits must be 1, 5 and 7, and the possibilities are 715 and 175. It is easy to check that neither of these works.

Next, if QED is odd *and* divisible by 3 then the possibilities for the digits are: 9, 5 and 1; or 9, 5 and 7; or 3, 5 and 1; or 3, 5 and 7. Divisibility by 9 rules out the first two possibilities, leaving us with 135, 315, 375 and 735. Checking, we see that 135 works but none of the others work.

Now, we want to rule out QED being even. Clearly, 5 cannot be one of the digits. And, the divisibility tricks show that there are only a few possibilities where 7 is one of the digits, and these are easily ruled out. Similarly, the divisibility rules help show that 3 and 9 can be ruled out as possible digits.

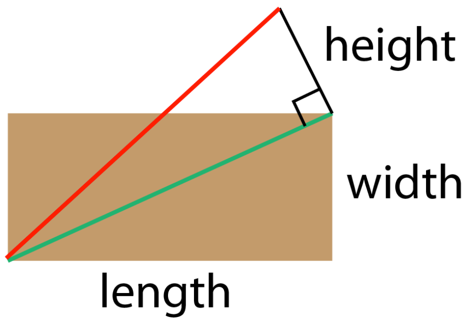
For QED even, that only leaves the possibility that all the digits are even, or that two of the digits are even with the third digit being 1. With all the digits even, the product is greater than 1000, unless the digits are 2, 4 and 6, or 2, 4 and 8; since QED is divisible by 4, the last digit cannot be 2 or 6, and there are only a few combinations to be ruled out.

Finally, if QED is even with two digits even and the third digit 1, then either Q = 1 and the last digit is 4 or 8, or E = 1 and the last digit is 2 or 6. This gives us just a few cases to consider, all of which can be ruled out.

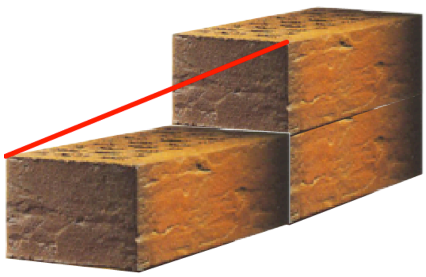
MEDIUM 7

You want to measure the longest diagonal of a brick. You have a long ruler, but you can't remember that formula by that Greek fellow! However, you do have a pen and paper. What do you do? What if you lose the pen and paper, but you now have three identical bricks?

ANSWER: In both cases, you want to make the long diagonal of the brick physically apparent. In the first case, place a brick on the paper and draw two sides of the brick. Use the ruler to draw in the missing (green) line, forming a triangle. Next, use the brick as a set square, to draw the third side of the brick at right angles to the triangle. Use the ruler to draw in and measure the missing (red) line of a second triangle. The length of this red side is the same as the length of the long diagonal of the brick.



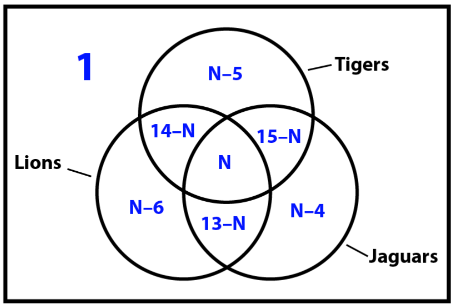
In the second case, stack the bricks as indicated to make the length of the long diagonal accessible.



MEDIUM 8

36 students went to the zoo. 21 students liked the lions, 24 students liked the tigers, and 24 students liked the jaguars. 14 students like the lions and the tigers, 15 students liked the tigers and the jaguars, and 13 students liked the jaguars and the lions. One grumpy student didn't like any of the animals. How many students liked the lions and the tigers and the jaguars?

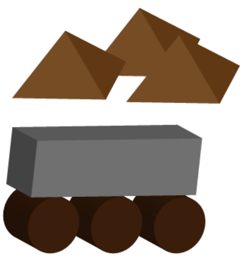
ANSWER: Suppose that N students like all of the animals. Then 14 – N students like the lions and tigers, but not the jaguars. Similarly, 15 – N students like the tigers and jaguars, but not the lions. Continuing in this manner, we can fill in the whole Venn diagram as indicated.



Since there are 36 students in total, adding up gives us the equation N + 28 = 36, and so 8 students liked all the animals.

MEDIUM 9

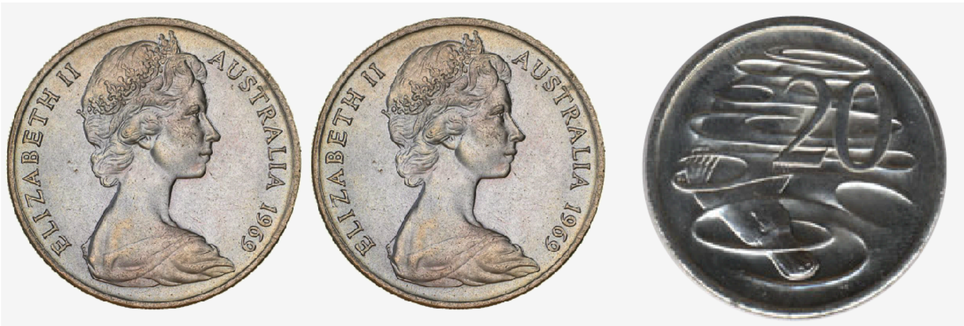
A stone to build the great pyramid of Giza is moved by rolling it on tree trunks, which are 2 meters in diameter? If the trunks make one complete revolution, how far does the stone move?



ANSWER: With one revolution, the centre of a trunk moves along 2π meters. However, at any moment in time, the top of the trunk (furthest from the ground) has twice the horizontal speed of the centre. So, the stone is also moving at twice the speed, and moves along 4π meters with one revolution.

MEDIUM 10

Brad and Angelina each have a handful of coins, with Angelina having one more coin than Brad. They both throw their coins on the table. What is the probability that Angelina obtains more heads than Brad?



ANSWER: There are four scenarios:

a) Angelina throws more heads than Brad, but not more tails;
b) Angelina throws more tails than Brad, but not more heads;
c) Angelina throws more heads and more tails than Brad;
d) Angelina throws neither more heads nor more tails than Brad.

However, the fourth scenario is impossible, since Angelina has more coins than Brad. And, the third scenario is impossible, since Angelina has only one more coin than Brad. That leaves only the two first scenarios, which are equally likely. So, the probability of each scenario is 1/2.